THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH3070 Introduction to Topology 2017-2018
Tutorial Classwork 0

1. For every set X, the cocountable topology is defined by $\mathfrak{T} = \{\emptyset, X\} \cup \{G \subset X \mid X \backslash G \text{ is countable}\}$.

   (a) Show that the $\mathfrak{T}$ is a topology.

   (b) Show that if X is uncountable, then the cocountable topology is NOT a metric topology.

   (c) How about if X is countable?

2. Let $(X, \mathfrak{T}_d)$ and $(X, \mathfrak{T}_\rho)$ be two topological spaces equipped with metrics d and ρ respectively. Show that if there exists $k > 0$ such that $d(x, y) \leq k\rho(x, y)$ for all $x, y \in X$, then $\mathfrak{T}_d \subset \mathfrak{T}_\rho$.

3. Let $X = (-\frac{\pi}{2}, \frac{\pi}{2})$ equipped with the standard metric $d(x, y) = |x - y|$. Define $\rho : X \times X \to [0, \infty)$ by $\rho(x, y) = |\tan x - \tan y|$.

   (a) Show that (X, ρ) is a metric space.

   (b) Show that the function $f : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ defined by $f(x) = \tan x - x$ is increasing. Hence, or otherwise, show that the metric topology $\mathfrak{T}_\rho$ coincides with the standard topology $\mathfrak{T}_d$.

   (c)* Recall that the set of real number $(\mathbb{R}, d)$ is *complete*, i.e. every Cauchy sequence converges. Show that (X, d) is incomplete while (X, ρ) is complete.

   (This question shows you that completeness is a property about metric but not topology.)

   (d)* Given a metric space (X, d) and an open set $A \subset X$. In general (A, d) may not be complete. Can you define a new metric τ such that $\mathfrak{T}_d = \mathfrak{T}_\tau$ and (A, τ) is complete?